Percolation thresholds of the Fortuin-Kasteleyn cluster for the Edwards-Anderson model on complex network: analytical results on the Nishimori Line

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We show analytically the percolation thresholds of the Fortuin-Kasteleyn cluster for the Edwards-Anderson model on random graphs with arbitary degree distributions. The results on the Nishimori line are shown. The results for the $\pm J$ model, the diluted $\pm J$ model, and the Gaussian model are respectively shown. The results are shown by applying an extension of a criterion for the random graphs with arbitary degree distributions. The results for the infinite-range $\pm J$ model and the Sherrington-Kirkpatrick model are also shown.

KEYWORDS: spin glass, percolation, complex network, the Nishimori line, cluster algorithm

1. Introduction

Study of spin models on complex network has been important[1] with the development of study of complex network. In the present article, random graphs with arbitary degree distributions are adopted as examples of the complex network. In the present article, behavior of spins on no growing network is investigated.

We investigate the Edwards-Anderson model[2] as an Ising spin-glass model. The understandings for the Edwards-Anderson model on random graphs and on the Bethe lattice are still uncompleted.[1,3,4] In the present article, the $\pm J$ model, the diluted $\pm J$ model, and the Gaussian model in the Edwards-Anderson model are respectively investigated. The Nishimori line in the Edwards-Anderson model is a line in the phase diagram for the exchange interactions and the temperature. The internal energy, the upper bound of the specific heat, and so forth are exactly calculated on the Nishimori line in the Edwards-Anderson model.[5–9] The location of the multicritical point in the Edwards-Anderson model on the square lattice is conjectured, and it is shown that the conjectured value is in good agreement with the other numerical estimates.[10] In the present article, the results on the Nishimori line are shown.

There is a case for occuring a percolation transition of network that the network is devided into a lot of networks by deleting nodes and/or links on a network. We define the percolation transition as the percolation transition of network in the present article. There is a case for occuring a percolation transition of cluster that the cluster consists of bonds put between spins becomes a giant component. The bond is fictitious. We define the percolation transition as the percolation transition of cluster in the present article. In the present article, the percolation transition of cluster on a complex network is mentioned.

In the present article, the percolation transition of the Fortuin-Kasteleyn (FK) cluster in the FK random cluster model[11] is investigated. In the ferromagnetic spin model, the percolation transition point of the FK cluster agrees with the phase transition point. The agreement gives the geometrical understanding of the phase transition phenomenon.[12] Powerful Monte Carlo methods using the FK cluster have been proposed.[13–17] On the other hand, in the Edwards-Anderson model that has a conflict in the interactions, the percolation transition point of the FK cluster disagrees with the phase transition point. The disagreement makes the geometrical understanding of the phase transition phenomenon awkward. There are plenty of approches for solving the awkwardness by using extensions of the FK random cluster model.[18] On the other hand, it is pointed out, by Arcangelis, A. Coniglio, and F. Peruggi, that the correct understanding of the percolation phenomenon of the FK cluster in the Edwards-Anderson model is important since a dynamical transition, which is characterized by a parameter called the Hamming distance or damage, is occurred at a temperature very close to the percolation temperature, and the dynamical transition and the percolation transition are related to a transition for a signal propagating between spins.[19] In the present article, the percolation threshold of the FK cluster in the Edwards-Anderson model is analytically shown.

Study of the random graphs with arbitary degree distributions has been developed.[20] Our results are shown by applying an extension of a criterion[21–23] for the random graphs with arbitary degree distributions. The results for the infinite-range $\pm J$ model and the results for the Sherrington-Kirkpatrick (SK) model[24] are also shown.

The present article is organised as follows. First, a complex network model and the Edwards-Anderson model are described. The FK cluster is described. A criterion for percolation of cluster is described. The percolation thresholds are shown for the $\pm J$ model and the diluted $\pm J$ model. The percolation threshold is shown for the Gaussian model. The present article is summarized.

2. A complex network model and the Edwards-Anderson model

Network consists of nodes and links connected between nodes. In the present article, the network models that we investigate are random graphs with arbitary degree distributions. The network has no correlation between nodes. The node degree, k, is generated according to a distribution of the node degree, $p(k)$. The links are ran-

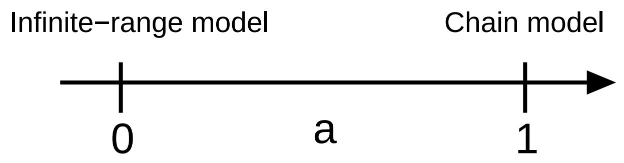

Fig. 1. A relation between the aspect a of the network and the model on the corresponding network of the aspect a.

domly connected between nodes.

We define a variable $b(i,j)$ where $b(i,j)$ gives one when node i and node j is connected by a link and gives zero when node i and node j is not connected by the link. The degree $k(i)$ of node i is given by

$$k(i) = \sum_j b(i,j)\,. \tag{1}$$

The average of the node degree for links, $< k >_N$, is given by

$$< k >_N = \frac{1}{N}\sum_i^N k(i) \tag{2}$$

where $<>_N$ is the average over the whole network. N is the number of nodes. The average of the square of the node degree for links, $< k^2 >_N$, is given by

$$< k^2 >_N = \frac{1}{N}\sum_i^N k^2(i)\,. \tag{3}$$

We define

$$a = \frac{2 < k >_N}{< k^2 >_N} \tag{4}$$

where a represents an aspect of the network.

Fig. 1 shows a relation between the aspect a of the network and the model on the corresponding network of the aspect a. A network is almost the complete graph when a is close to zero, and the model on the network is almost an infinite-range model. The model on the network is the infinite-range model when $< k >_N = N-1$, $< k^2 >_N = (N-1)^2$, and $a = 2/(N-1)$. We define a coordination number as z. $p(k) = \delta_{k,z}$ where δ is the Dirac delta function. The coordination number z is two when the aspect a of the network is one. The network consists of many cycle graphs when z is two. The model on the network consists of many chain models. In the Erdős-Rényi (ER) random graph model and in the Gilbert model, the distribution of node degree is the Poisson distribution.[1] The ER random graph model is a network model that the network consists of the fixed number of nodes and the fixed number of links, and the links are randomly connected between the nodes. The Gilber model is a network model that link between nodes is connected with a given probability. In the ER random graph model and in the Gilbert model, $< k >_N = 1$ and $< k^2 >_N = < k >_N (< k >_N + 1) = 2$ when a is one.

The Hamiltonian for the Edwards-Anderson model, $\mathcal{H}$, is given by

$$\mathcal{H} = -\frac{1}{2}\sum_i^N \sum_{\{j|b(i,j)=1\}} J_{ij} S_i S_j \tag{5}$$

where $S_i = \pm 1$. J_{ij} is the strength of the exchange interaction between the spin on node i and the spin on node j. The value of J_{ij} is given with the distribution $P(J_{ij})$. The $\pm J$ model, the diluted $\pm J$ model, and the Gaussian model are respectively given with the difference of $P(J_{ij})$.

For the $\pm J$ model, the distribution $P^{(\pm J)}(J_{ij})$ of J_{ij} is given by

$$P^{(\pm J)}(J_{ij}) = p\,\delta_{J_{ij},J} + (1-p)\,\delta_{J_{ij},-J} \tag{6}$$

where $J > 0$. p is the probability that the interaction is the ferromagnetic interaction ($J_{ij} = J$). $1-p$ is the probability that the interaction is the antiferromagnetic interaction ($J_{ij} = -J$).

For the diluted $\pm J$ model, the distribution $P^{(\mathrm{D}\pm J)}(J_{ij})$ of J_{ij} is given by

$$P^{(\mathrm{D}\pm J)}(J_{ij}) = p\,\delta_{J_{ij},J} + q\,\delta_{J_{ij},-J} + r\,\delta_{J_{ij},0} \tag{7}$$

where $J > 0$ and $p+q+r = 1$. p is the probability that the interaction is the ferromagnetic interaction ($J_{ij} = J$). q is the probability that the interaction is the antiferromagnetic interaction ($J_{ij} = -J$). r is the probability that the interaction is diluted ($J_{ij} = 0$). This model is the $\pm J$ model when $r = 0$.

For the Gaussian model, the distribution $P^{(\mathrm{Gaussian})}(J_{ij})$ of J_{ij} is given by

$$P^{(\mathrm{Gaussian})}(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} e^{-(J_{ij}-J_0)^2/2J^2}\,. \tag{8}$$

The average of J_{ij} is $[J_{ij}]_R = J_0$ where $[]_R$ is the random configuration average. The variance of J_{ij} is $[J_{ij}^2]_R - [J_{ij}]_R^2 = J^2$.

In the present article, for calculating thermodynamic quantities, a gauge transformation:

$$J_{ij} \to J_{ij}\sigma_i\sigma_j\,, \quad S_i \to S_i\sigma_i \tag{9}$$

is used where $\sigma_i = \pm 1$. Using the gauge transformation, $\mathcal{H} \to \mathcal{H}$ and $P(J_{ij}) \to P(J_{ij}\sigma_i\sigma_j)$. This transformation has no effect on thermodynamic quantities.[25]

3. The Fortuin-Kasteleyn cluster

The bond for the FK cluster is put between spins with the corresponding probability $P_{\mathrm{bond}}(S_i, S_j, J_{ij})$ of the spin states and the strength of the exchange interaction. The probability $P_{\mathrm{bond}}(S_i, S_j, J_{ij})$ is given by

$$P_{\mathrm{bond}}(S_i, S_j, J_{ij}) = 1 - e^{-\beta J_{ij} S_i S_j - \beta |J_{ij}|} \tag{10}$$

[19] where β is the inverse temperature and $\beta = 1/k_B T$. k_B is the Boltzmann constant and T is the temperature. By connecting the bonds, the FK clusters are generated.

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[< b_{\mathrm{bond}}(i,j) >_T]_R$, is

$$[< b_{\mathrm{bond}}(i,j) >_T]_R = [< P_{\mathrm{bond}}(S_i, S_j, J_{ij}) >_T]_R \tag{11}$$

where $<>_T$ is the thermal average. The thermody-

namic quantity of the node degree for bonds at node i, $[< k_{\mathrm{bond}}(i) >_T]_R$, is

$$[< k_{\mathrm{bond}}(i) >_T]_R = [< \sum_{\{j|b(i,j)=1\}} P_{\mathrm{bond}}(S_i, S_j, J_{ij}) >_T]_R \,. \quad (12)$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k^2_{\mathrm{bond}}(i) >_T]_R$, is

$$[< k^2_{\mathrm{bond}}(i) >_T]_R = [< \sum_{\{j|b(i,j)=1\}} \sum_{\{l|b(i,l)=1\}} P_{\mathrm{bond}}(S_i, S_j, J_{ij}) \times P_{\mathrm{bond}}(S_i, S_l, J_{il})(1 - \delta_{j,l}) + \sum_{\{j|b(i,j)=1\}} P_{\mathrm{bond}}(S_i, S_j, J_{ij}) >_T]_R \,. \quad (13)$$

The thermodynamic quantity of the node degree for bonds, $[< k_{\mathrm{bond}} >_T]_R$, is

$$[< k_{\mathrm{bond}} >_T]_R = \frac{1}{N} \sum_i^N [< k_{\mathrm{bond}}(i) >_T]_R \,. \quad (14)$$

The thermodynamic quantity of the square of the node degree for bonds, $[< k^2_{\mathrm{bond}} >_T]_R$, is

$$[< k^2_{\mathrm{bond}} >_T]_R = \frac{1}{N} \sum_i^N [< k^2_{\mathrm{bond}}(i) >_T]_R \,. \quad (15)$$

4. A criterion for percolation of cluster

The percolation of the random graphs with arbitary degree distributions is occured when

$$< k^2 >_N \geq 2 < k >_N \quad (16)$$

.[21–23] Ineq. (16) is given by the inequality when the network is percolated. Ineq. (16) is given by the equality when the network is at the percolation transition point. The criterion (ineq. (16)) is true for sufficiently large number of nodes. Ineq. (16) is derived, with various methods, by the group of Molly and Reed,[21] the group of Cohen et. al.,[22] and the group of Newman et. al.,[23] respectively.

From ineq. (16), the network is percolated when $a < 1$. From ineq. (16), the network is at the percolation transition point when $a = 1$. The network and the cluster on the network are unpercolated when $a > 1$. Therefore, the percolation of cluster is investigated for $0 < a \leq 1$.

In the case that links and/or nodes are randomly diluted on the random graphs with arbitary degree distributions, the criterion (ineq. (16)) is applicable to the diluted network.[22] In the case that links are diluted on the random graphs with arbitary degree distributions, the percolation problem for the diluted network can be regarded as the bond percolation problem. We define the bond states as the graph G. In the graph G, we define the node degree for bonds at node i as $k_{\mathrm{bond}}(G, i)$. Since the bonds are randomly put on the links that are randomly connected between the nodes, the criterion of the percolation of cluster for the bond percolation problem is

$$\frac{1}{N} \sum_i k^2_{\mathrm{bond}}(G, i) \geq \frac{2}{N} \sum_i k_{\mathrm{bond}}(G, i) \,. \quad (17)$$

In what follows, a criterion of the percolation of cluster for spin models is given based on the above discussion.

We consider a condition that the bond does not depend on the size of the degree $k(i)$. We define a variable for the inverse temperature as $\rho(\beta)$. We set

$$0 < \rho(\beta) \leq 1 \,. \quad (18)$$

We assume the case that $[< b_{\mathrm{bond}}(i,j) >_T]_R$, $[< k_{\mathrm{bond}}(i) >_T]_R$, and $[< k^2_{\mathrm{bond}}(i) >_T]_R$ are respectively written in

$$[< b_{\mathrm{bond}}(i,j) >_T]_R = \rho(\beta) \,, \quad (19)$$

$$[< k_{\mathrm{bond}}(i) >_T]_R = \rho(\beta)\, k(i) \,, \quad (20)$$

$$[< k^2_{\mathrm{bond}}(i) >_T]_R = \rho^2(\beta)\, k(i)[k(i) - 1] + \rho(\beta)\, k(i) \,. \quad (21)$$

In the case, it is implied that the bias for the size of $k(i)$ does not appear in the statistical results of bonds put between spins. Therefore, we describe the case that $[< b_{\mathrm{bond}}(i,j) >_T]_R$, $[< k_{\mathrm{bond}}(i) >_T]_R$, and $[< k^2_{\mathrm{bond}}(i) >_T]_R$ are respectively written in eqs. (19, 20, 21) as the case that the condition that the bond does not depend on the size of $k(i)$ is satisfied.

In the case that the condition is satisfied, as an extension of ineq. (17), we obtain

$$\frac{1}{N} \sum_i k^2_{\mathrm{bond}}(\{S_j\}, \{J_{jl}\}, G, i) \geq \frac{2}{N} \sum_i k_{\mathrm{bond}}(\{S_j\}, \{J_{jl}\}, G, i) \quad (22)$$

since the bonds that do not depend on the size of $k(i)$ are put on the links that are randomly connected between nodes. $k_{\mathrm{bond}}(\{S_j\}, \{J_{jl}\}, G, i)$ is the node degree for bonds at node i in the graph G that is compatible of $\{S_j\}$ and $\{J_{jl}\}$. It is anticipated that ineq. (22) is true for sufficiently large number of nodes in the case that the condition is satisfied. Using ineq. (22), we obtain the criterion of the percolation of cluster for spin models as

$$[< k^2_{\mathrm{bond}} >_T]_R \geq 2[< k_{\mathrm{bond}} >_T]_R \,. \quad (23)$$

Ineq. (23) is given by the inequality when the cluster is percolated. Ineq. (23) is given by the equality when the cluster is at the percolation transition point. Ineq. (23) gives the percolation threshold of the cluster in the case that the condition is satisfied.

5. The $\pm J$ model and the dilued $\pm J$ model

The distribution $P^{(\pm J)}(J_{ij})$ of J_{ij} for the $\pm J$ model is, using eq. (6),

$$P^{(\pm J)}(J_{ij}) = \frac{e^{\beta_{\mathrm{P}} J_{ij}}}{2\cosh(\beta_P J)} \,, \quad J_{ij} = \pm J \quad (24)$$

where β_P is given by

$$\beta_P = \frac{1}{2J} \ln \frac{p}{1-p} \tag{25}$$

.[5]

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[< b_{\rm bond}(i,j) >_T]_R^{(\pm J)}$, is, using eqs. (9, 10, 11, 24) when $\beta = \beta_P$,

$$\begin{aligned}
& [< b_{\rm bond}(i,j) >_T]_R^{(\pm J)} \\
= & \sum_{\{J_{lm}\}} \prod_{<lm>} P^{(\pm J)}(J_{lm}) \times \\
& \frac{\sum_{\{S_l\}} P_{\rm bond}(S_i, S_j, J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}} \\
= & \frac{1}{2^N [2\cosh(\beta_P J)]^{N_B}} \times \\
& \sum_{\{J_{lm}\}} \sum_{\{S_l\}} P_{\rm bond}(S_i, S_j, J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m} \\
= & \tanh(\beta_P J)
\end{aligned} \tag{26}$$

where $< xy >$ denotes the nearest neighbor pairs connected by links, N_B is the number of all links, and $N_B = N < k >_N /2$. The thermodynamic quantity of the node degree for bonds at node i, $[< k_{\rm bond}(i) >_T]_R^{(\pm J)}$, is, using eqs. (9, 10, 12, 24) when $\beta = \beta_P$,

$$[< k_{\rm bond}(i) >_T]_R^{(\pm J)} = \tanh(\beta_P J)\, k(i)\,. \tag{27}$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k_{\rm bond}^2(i) >_T]_R^{(\pm J)}$, is, using eqs. (9, 10, 13, 24) when $\beta = \beta_P$,

$$\begin{aligned}
[< k_{\rm bond}^2(i) >_T]_R^{(\pm J)} = & \tanh^2(\beta_P J)\, k(i)[k(i)-1] + \\
& \tanh(\beta_P J)\, k(i)\,.
\end{aligned} \tag{28}$$

We set

$$\rho^{(\pm J)}(\beta_P) = \tanh(\beta_P J)\,. \tag{29}$$

Eqs. (26, 27, 28, 29) are formulated as eqs. (19, 20, 21), respectively. Therefore, the bond does not depend on the size of $k(i)$. Using eqs. (14, 15, 23, 27, 28), we obtain the criterion of the percolation of the FK cluster for the $\pm J$ model as

$$1 - \exp(-2\beta_P J) \geq \frac{2 < k >_N}{< k^2 >_N}\,. \tag{30}$$

Ineq. (30) is given by the inequality when the cluster is percolated. Ineq. (30) is given by the equality when the cluster is at the percolation transition point.

From eqs. (18, 29), it is realized that there is the percolation transition point for $0 < \beta_P \leq \infty$. From eq. (30), it is realized that there is the percolation transition point for $0 < a \leq 1$. The probability $p^{(\pm J)}$ that the interaction is the ferromagnetic interaction is, using eqs. (25, 30),

$$p^{(\pm J)} = \frac{1}{2-a} \tag{31}$$

at the percolation transition point. The percolation transition temperature $T_P^{(\pm J)}$ is, using eqs. (25, 31),

$$T_P^{(\pm J)} = \frac{J}{k_B} \frac{2}{\ln \frac{1}{1-a}}\,. \tag{32}$$

In Ref.,[5] the internal energy on the Nishimori line for the $\pm J$ model, $E^{(\pm J)}$, is derived. The internal energy $E^{(\pm J)}$ is equivalently

$$E^{(\pm J)} = -\frac{NJ\tanh(\beta_P J)}{2} < k >_N\ . \tag{33}$$

The internal energy $E^{(\pm J)}$ is, using eqs. (32, 33),

$$E^{(\pm J)} = -\frac{NaJ}{2(2-a)} < k >_N\ . \tag{34}$$

We define the specific heat on the Nishimori line for the $\pm J$ model as $C^{(\pm J)}$. In Refs.,[5,6] the upper bound of the specific heat $C^{(\pm J)}$ is derived. The upper bound of the specific heat $C^{(\pm J)}$ is equivalently

$$C^{(\pm J)} \leq \frac{k_B N (\beta_P J)^2 {\rm sech}^2(\beta_P J)}{2} < k >_N\ . \tag{35}$$

The upper bound of the specific heat $C^{(\pm J)}$ is, using eqs. (32, 35),

$$C^{(\pm J)} \leq \frac{k_B N (1-a)}{2(2-a)^2} \left(\ln \frac{1}{1-a} \right)^2 < k >_N\ . \tag{36}$$

It is realized that the specific heat has no singularity.

In Fig. 2, the percolation threshold of the FK cluster for the $\pm J$ model is shown. In (a) of Fig. 2, the relation between the aspect a of the network and the probability $p^{(\pm J)}$ that the interaction is the ferromagnetic interaction is shown. Eq. (31) is used for showing (a) of Fig. 2. In (b) of Fig. 2, the relation between the aspect a of the network and the percolation transition temperature $T_P^{(\pm J)}$ is shown. Eq. (32) is used for showing (b) of Fig. 2. In (b) of Fig. 2, $J/k_B = 1$ is set.

For the ferromagnetic Ising model on the same network, the phase transition temperature $T_C^{(\rm Ferro)}$ is

$$T_C^{(\rm Ferro)} = \frac{J}{k_B} \frac{2}{\ln \frac{1}{1-a}} \tag{37}$$

.[26,27] $T_P^{(\pm J)}$ (eq. (32)) coincides with $T_C^{(\rm Ferro)}$. For the ferromagnetic Ising model on the same network, the specific heat diverges or jumps at $T_C^{(\rm Ferro)}$.[26,27] On the other hand, from eq. (36), the specfic heat neither diverge nor jump at $T_P^{(\pm J)}$ on the Nishimori line.

The complete graph is considerd as a case that $a \sim 0$. We set $< k >_N = N-1$, $< k^2 >_N = (N-1)^2$, $a = 2/(N-1)$, and $J \to J/\sqrt{N}$. From the settings, the model on the network becomes the infinit-range $\pm J$ model. In what follows, our results for the infinit-range $\pm J$ model are shown. The probability $p^{({\rm IR}\pm J)}$ that the interaction is the ferromagnetic interaction is, using eq. (31) for sufficiently large number of nodes,

$$p^{({\rm IR}\pm J)} = \frac{N-1}{2(N-2)} \to \frac{1}{2} \tag{38}$$

at the percolation transition point. The percolation transition temperature $T_P^{({\rm IR}\pm J)}$ is, using eq. (32) for suffi-

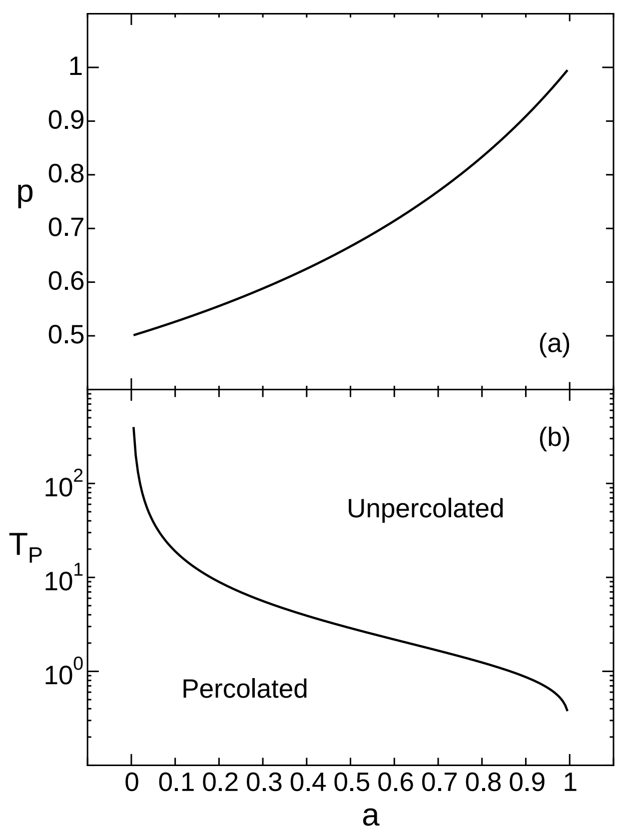

Fig. 2. The percolation threshold of the FK cluster for the $\pm J$ model. (a) The relation between the aspect a of the network and the probability $p^{(\pm J)}$ that the interaction is the ferromagnetic interaction is shown. (b) The relation between the aspect a of the network and the percolation transition temperature $T_P^{(\pm J)}$ is shown. $J/k_B = 1$ is set.

ciently large number of nodes,

$$T_P^{(\mathrm{IR}\pm J)} = \frac{J}{k_B\sqrt{N}} \frac{2}{\ln(1+\frac{2}{N-3})} \to \frac{J}{k_B}\sqrt{N}\,. \tag{39}$$

The internal energy $E^{(\mathrm{IR}\pm J)}$ is, using eq. (34) for sufficiently large number of nodes,

$$E^{(\mathrm{IR}\pm J)} = -\frac{\sqrt{N}(N-1)}{2(N-2)}J \to -\frac{J}{2}\sqrt{N}\,. \tag{40}$$

For sufficiently large number of nodes, the right-hand side of eq. (36) reduces to

$$\frac{k_B N(N-1)^2(N-3)}{8(N-2)^2}\left[\ln\left(1+\frac{2}{N-3}\right)\right]^2 \to \frac{k_B}{2}\,. \tag{41}$$

The upper bound of the specific heat $C^{(\mathrm{IR}\pm J)}$ is, using eqs. (36, 41),

$$C^{(\mathrm{IR}\pm J)} \leq \frac{k_B}{2}\,. \tag{42}$$

In Ref.,[18] the percolation transition temperature of the FK cluster for the infinit-range $\pm J$ model is derived by using the analytical solution of the SK model. The percolation transition temperature of the FK cluster for the infinit-range $\pm J$ model obtained in the present article agrees with the result for single replica case in Ref.[18]

The case for $a = 1$ is described. Using eq. (31), $p = 1$. Using eq. (32), $T_P = 0$. From eq. (16), it is realized that the network is at the percolation transition point. From $p = 1$, the exchange interaction is only the ferromagnetic interaction. From $T_P = 0$, all the spins are parallel. Therefore, the cluster and the network are at the percolation transition point since the bonds are put on all the links of the network.

The distribution $P^{(\mathrm{D}\pm J)}(J_{ij})$ of J_{ij} for the diluted $\pm J$ model is, using eq. (7),

$$P^{(\mathrm{D}\pm J)}(J_{ij}) = \frac{e^{\beta_P^{(2)} J_{ij}^2 + \beta_P J_{ij}}}{e^{\beta_P^{(2)} J^2 + \beta_P J} + 1 + e^{\beta_P^{(2)} J^2 - \beta_P J}} \tag{43}$$

where $\beta_P^{(2)}$ and β_P are respectively

$$\beta_P^{(2)} = \frac{1}{J^2}\ln\sqrt{\frac{pq}{r^2}}\,, \quad \beta_P = \frac{1}{J}\ln\sqrt{\frac{p}{q}} \tag{44}$$

.[9] This model becomes the $\pm J$ model when $r = 0$. In what follows, the result for $r \neq 0$ is only described since the result for the $\pm J$ model is already desicribed above.

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[< b_{\mathrm{bond}}(i,j) >_T]_R^{(\mathrm{D}\pm J)}$, is, using eqs. (9, 10, 11, 43) when $\beta = \beta_P$,

$$\begin{aligned}
& [< b_{\mathrm{bond}}(i,j) >_T]_R^{(\mathrm{D}\pm J)} \\
= \ & \sum_{\{J_{lm}\}} \prod_{<lm>} P^{(\mathrm{D}\pm J)}(J_{lm}) \times \\
& \frac{\sum_{\{S_l\}} P_{\mathrm{bond}}(S_i,S_j,J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm}S_lS_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{<lm>} J_{lm}S_lS_m}} \\
= \ & \frac{1}{2^N(e^{\beta_P^{(2)} J^2 + \beta_P J} + 1 + e^{\beta_P^{(2)} J^2 - \beta_P J})^{N_B}} \times \\
& \sum_{\{J_{lm}\}} \sum_{\{S_l\}} P_{\mathrm{bond}}(S_i,S_j,J_{ij}) \times \\
& e^{\beta_P^{(2)} \sum_{<lm>} J_{lm}^2 + \beta_P \sum_{<lm>} J_{lm}S_lS_m} \\
= \ & (1-r)\tanh(\beta_P J)\,.
\end{aligned} \tag{45}$$

The thermodynamic quantity of the node degree for bonds at node i, $[< k_{\mathrm{bond}}(i) >_T]_R^{(\mathrm{D}\pm J)}$, is, using eqs. (9, 10, 12, 43) when $\beta = \beta_P$,

$$[< k_{\mathrm{bond}}(i) >_T]_R^{(\mathrm{D}\pm J)} = (1-r)\tanh(\beta_P J)\, k(i)\,. \tag{46}$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k_{\mathrm{bond}}^2(i) >_T]_R^{(\mathrm{D}\pm J)}$, is, using eqs. (9, 10, 13, 43) when $\beta = \beta_P$,

$$\begin{aligned}
& [< k_{\mathrm{bond}}^2(i) >_T]_R^{(\mathrm{D}\pm J)} = \\
& (1-r)^2\tanh^2(\beta_P J)\, k(i)[k(i)-1] + \\
& (1-r)\tanh(\beta_P J)\, k(i)\,.
\end{aligned} \tag{47}$$

We set

$$\rho^{(\mathrm{D}\pm J)}(\beta_P) = (1-r)\tanh(\beta_P J)\,. \tag{48}$$

Eqs. (45, 46, 47, 48) are formulated as eqs. (19, 20, 21), respectively. Therefore, the bond does not depend on the

size of $k(i)$. Using eqs. (14, 15, 23, 46, 47), we obtain the criterion of the percolation of the FK cluster for the diluted $\pm J$ model as

$$\frac{2(1-r)(1-e^{-2\beta_P J})}{1+e^{-2\beta_P J}+(1-r)(1-e^{-2\beta_P J})} \geq \frac{2<k>_N}{<k^2>_N} . \quad (49)$$

Ineq. (49) is given by the inequality when the cluster is percolated. Ineq. (49) is given by the equality when the cluster is at the percolation transition point.

From eqs. (18, 48), it is realized that there is the percolation transition point for $r \neq 1$ and $0 < \beta_P \leq \infty$. Using ineq. (49),

$$\frac{(2-a)(1-r)-a}{(2-a)(1-r)+a} \geq e^{-2\beta_P J} \geq 0 . \quad (50)$$

Using the left-hand side of ineq. (50) and the right-hand side of ineq. (50),

$$1-r \geq \frac{a}{2-a} . \quad (51)$$

When ineq. (51) is satisfied, there is the percolation transition point.

The probability $p^{(\mathrm{D}\pm J)}$ that the interaction is the ferromagnetic interaction is, using eqs. (44, 49),

$$p^{(\mathrm{D}\pm J)} = \frac{(2-a)(1-r)+a}{2(2-a)(1-r)} \quad (52)$$

at the percolation transition point. The percolation transition temperature $T_P^{(\mathrm{D}\pm J)}$ is, using eqs. (44, 52),

$$T_P^{(\mathrm{D}\pm J)} = \frac{J}{k_B} \frac{2}{\ln \frac{(2-a)(1-r)+a}{(2-a)(1-r)-a}} . \quad (53)$$

In Ref.,[9] the internal energy on the Nishimori line for the diluted $\pm J$ model, $E^{(\mathrm{D}\pm J)}$, is derived. The internal energy $E^{(\mathrm{D}\pm J)}$ is equivalently

$$E^{(\mathrm{D}\pm J)} = -\frac{NJ(1-r)\tanh(\beta_P J)}{2} <k>_N . \quad (54)$$

The internal energy $E^{(\mathrm{D}\pm J)}$ is, using eqs. (53, 54),

$$E^{(\mathrm{D}\pm J)} = -\frac{NaJ}{2(2-a)} <k>_N . \quad (55)$$

We define the specific heat on the Nishimori line for the diluted $\pm J$ model as $C^{(\mathrm{D}\pm J)}$. In Ref.,[9] the upper bound of the specific heat $C^{(\mathrm{D}\pm J)}$ is derived. The upper bound of the specific heat $C^{(\mathrm{D}\pm J)}$ is equivalently

$$\begin{aligned} C^{(\mathrm{D}\pm J)} &\leq \frac{k_B N(\beta_P J)^2}{2} <k>_N \times \\ & [(1-r)^2 \mathrm{sech}^2(\beta_P J) + r(1-r)] . \quad (56) \end{aligned}$$

The upper bound of the specific heat $C^{(\mathrm{D}\pm J)}$ is, using eqs. (53, 56),

$$\begin{aligned} C^{(\mathrm{D}\pm J)} &\leq \frac{k_B N}{8}\left[1-r-\frac{a^2}{(2-a)^2}\right] \times \\ & \left[\ln \frac{(2-a)(1-r)+a}{(2-a)(1-r)-a}\right]^2 <k>_N . \quad (57) \end{aligned}$$

The upper bound of the specific heat diverges when $r = 2(1-a)/(2-a)$. Using eq. (52), $p = 1$ when $r = 2(1-a)/(2-a)$. Using eq. (53), $T_P = 0$ when $r = 2(1-a)/(2-a)$. Using ineqs. (51, 57),

$$1-r-\frac{a^2}{(2-a)^2} \geq \frac{2a(1-a)}{(2-a)^2} . \quad (58)$$

The upper bound of the specific heat $C^{(\mathrm{D}\pm J)}$ is nonnegative.

6. The Gaussian model

The distribution $P^{(\mathrm{Gaussian})}(J_{ij})$ of J_{ij} for the Gaussian model is given in eq. (8). We set

$$\beta_P = \frac{J_0}{J^2} \quad (59)$$

.[5]

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[<b_{\mathrm{bond}}(i,j)>_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (8, 9, 10, 11, 59) when $\beta = \beta_P$,

$$\begin{aligned} & [<b_{\mathrm{bond}}(i,j)>_T]_R^{(\mathrm{Gaussian})} \\ = & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{<lm>} dJ_{lm} \prod_{<lm>} P^{(\mathrm{Gaussian})}(J_{lm}) \times \\ & \frac{\sum_{\{S_l\}} P_{\mathrm{bond}}(S_i,S_j,J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}} \\ = & \frac{1}{2^N (2\pi J^2)^{N_B/2}} e^{-N_B \frac{J_0^2}{2J^2}} \times \\ & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{<lm>} dJ_{lm} \sum_{\{S_l\}} P_{\mathrm{bond}}(S_i,S_j,J_{ij}) \times \\ & e^{-\sum_{<lm>} \frac{J_{lm}^2}{2J^2} + \beta_P \sum_{<lm>} J_{lm} S_l S_m} \\ = & \operatorname{erf}(\beta_P J/\sqrt{2}) \quad (60) \end{aligned}$$

where $\operatorname{erf}(x)$ is the error function of x. The thermodynamic quantity of the node degree for bonds at node i, $[<k_{\mathrm{bond}}(i)>_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (8, 9, 10, 12, 59) when $\beta = \beta_P$,

$$[<k_{\mathrm{bond}}(i)>_T]_R^{(\mathrm{Gaussian})} = \operatorname{erf}(\beta_P J/\sqrt{2})\, k(i) . \quad (61)$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[<k_{\mathrm{bond}}^2(i)>_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (8, 9, 10, 13, 59) when $\beta = \beta_P$,

$$\begin{aligned} & [<k_{\mathrm{bond}}^2(i)>_T]_R^{(\mathrm{Gaussian})} = \\ & [\operatorname{erf}(\beta_P J/\sqrt{2})]^2 k(i)[k(i)-1] + \\ & \operatorname{erf}(\beta_P J/\sqrt{2})\, k(i) . \quad (62) \end{aligned}$$

We set

$$\rho^{(\mathrm{Gaussian})}(\beta_P) = \operatorname{erf}(\beta_P J/\sqrt{2}) . \quad (63)$$

Eqs. (60, 61, 62, 63) are formulated as eqs. (19, 20, 21), respectively. Therefore, the bond does not depend on the size of $k(i)$. Using eqs. (14, 15, 23, 61, 62), we obtain the criterion of the percolation of the FK cluster for the Gaussian model as

$$\frac{2\operatorname{erf}(\beta_P J/\sqrt{2})}{\operatorname{erf}(\beta_P J/\sqrt{2})+1} \geq \frac{2<k>_N}{<k^2>_N} . \quad (64)$$

Ineq. (64) is given by the inequality when the cluster is percolated. Ineq. (64) is given by the equality when the cluster is at the percolation transition point.

From eqs. (18, 63), it is realized that there is the percolation transition point for $0 < \beta_P \leq \infty$. From eq. (64), it is realized that there is the percolation transition point for $0 < a \leq 1$. We approximate the error function $\mathrm{erf}(x)$ by

$$\mathrm{erf}(x) \approx \sqrt{1 - \exp(-4x^2/\pi)}\,. \tag{65}$$

J_0/J is, using eqs. (59, 64, 65),

$$\frac{J_0}{J} = \sqrt{\frac{\pi}{2} \ln \frac{(2-a)^2}{4(1-a)}} \tag{66}$$

at the percolation transition point. The percolation transition temperature $T_P^{(\mathrm{Gaussian})}$ is, using eqs. (59, 66),

$$T_P^{(\mathrm{Gaussian})} = \frac{J}{k_B} \frac{1}{\sqrt{\frac{\pi}{2} \ln \frac{(2-a)^2}{4(1-a)}}}\,. \tag{67}$$

In Ref.,[5] the internal energy on the Nishimori line for the Gaussian model, $E^{(\mathrm{Gaussian})}$, is derived. The internal energy $E^{(\mathrm{Gaussian})}$ is equivalently

$$E^{(\mathrm{Gaussian})} = -\frac{NJ_0}{2} < k >_N \ . \tag{68}$$

The internal energy $E^{(\mathrm{Gaussian})}$ is, using eqs. (66, 68),

$$E^{(\mathrm{Gaussian})} = -\frac{NJ}{2} \sqrt{\frac{\pi}{2} \ln \frac{(2-a)^2}{4(1-a)}} < k >_N \ . \tag{69}$$

We define the specific heat on the Nishimori line for the Gaussian model as $C^{(\mathrm{Gaussian})}$. In Ref.,[5] the upper bound of the specific heat $C^{(\mathrm{Gaussian})}$ is derived. The upper bound of the specific heat $C^{(\mathrm{Gaussian})}$ is equivalently

$$C^{(\mathrm{Gaussian})} \leq \frac{k_B N (\beta_P J)^2}{2} < k >_N \ . \tag{70}$$

The upper bound of the specific heat $C^{(\mathrm{Gaussian})}$ is, using eqs. (67, 70),

$$C^{(\mathrm{Gaussian})} \leq \frac{k_B N \pi}{4} \ln\left[\frac{(2-a)^2}{4(1-a)}\right] < k >_N \ . \tag{71}$$

In Fig. 3, the percolation threshold of the FK cluster for the Gaussian model is shown. In (a) of Fig. 3, the relation between the aspect a of the network and J_0/J is shown. Eq. (66) is used for showing (a) of Fig. 3. In (b) of Fig. 3, the relation between the aspect a of the network and the percolation transition temperature $T_P^{(\mathrm{Gaussian})}$ is shown. Eq. (67) is used for showing (b) of Fig. 3. In (b) of Fig. 3, $J/k_B = 1$ is set.

The complete graph is considerd as a case that $a \sim 0$. We set $< k >_N = N-1$, $< k^2 >_N = (N-1)^2$, $a = 2/(N-1)$, $J_0 \to J_0/N$, and $J \to J/\sqrt{N}$. From the settings, the model on the network becomes the SK model.[24] In what follows, our results for the SK model are shown. J_0/J is, using eq. (66) for sufficiently large number of nodes,

$$\frac{J_0}{J} = \sqrt{\frac{\pi N}{2} \ln\left(1 + \frac{1}{N^2 - 4N + 3}\right)} \to \sqrt{\frac{\pi}{2N}} \tag{72}$$

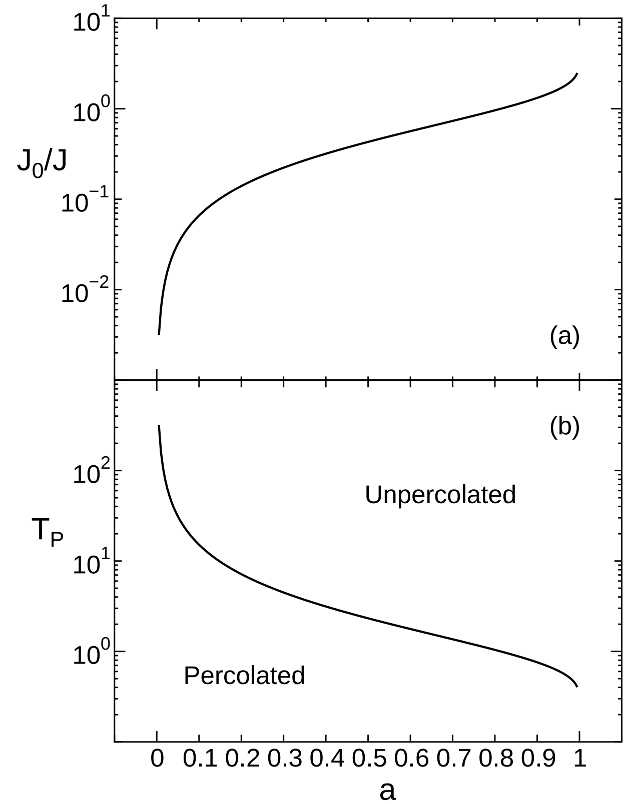

Fig. 3. The percolation threshold of the FK cluster for the Gaussian model. (a) The relation between the aspect a of the network and J_0/J is shown. (b) The relation between the aspect a of the network and the percolation transition temperature $T_P^{(\mathrm{Gaussian})}$ is shown. $J/k_B = 1$ is set.

at the percolation transition point. The percolation transition temperature $T_P^{(\mathrm{SK})}$ is, using eq. (67) for sufficiently large number of nodes,

$$\begin{aligned} T_P^{(\mathrm{SK})} &= \frac{J}{k_B} \frac{1}{\sqrt{\frac{\pi N}{2} \ln\left(1 + \frac{1}{N^2-4N+3}\right)}} \\ &\to \frac{J}{k_B} \sqrt{\frac{2N}{\pi}}\,. \end{aligned} \tag{73}$$

The internal energy $E^{(\mathrm{SK})}$ is, using eq. (69) for sufficiently large number of nodes,

$$\begin{aligned} E^{(\mathrm{SK})} &= -\frac{J(N-1)}{2} \sqrt{\frac{\pi N}{2} \ln\left(1 + \frac{1}{N^2 - 4N + 3}\right)} \\ &\to -J\sqrt{\frac{\pi N}{8}}\,. \end{aligned} \tag{74}$$

For sufficiently large number of nodes, the right-hand side of eq. (71) reduces to

$$\frac{k_B N (N-1)\pi}{4} \ln\left(1 + \frac{1}{N^2 - 4N + 3}\right) \to \frac{k_B \pi}{4}\,. \tag{75}$$

The upper bound of the specific heat $C^{(\mathrm{SK})}$ is, using

eqs. (71, 75),

$$C^{(\mathrm{SK})} \leq \frac{k_B \pi}{4} \,. \tag{76}$$

The case for $a = 1$ is described. Using eq. (66), $J_0/J = \infty$. Using eq. (67), $T_P = 0$. From eq. (16), it is realized that the network is at the percolation transition point. From $J_0/J = \infty$, the exchange interaction is only the ferromagnetic interaction. From $T_P = 0$, all the spins are parallel. Therefore, the cluster and the network are at the percolation transition point since the bonds are put on all the links of the network.

In the result for the Gaussian model, an approximation formula of the error function, eq. (65), is used. In the result for the Gaussian model, it is necessary for the more precise estimation of the percolation threshold that the error function in eq. (64) is numerically estimated.

7. Summary

In the present article, the $\pm J$ model, the diluted $\pm J$ model, and the Gaussian model on the random graphs with artibary degree distributions are investigated.

In the present article, for the bond that generates the FK cluster, $[< b_{\mathrm{bond}}(i,j) >_T]_R$, $[< k_{\mathrm{bond}}(i) >_T]_R$, $[< k^2_{\mathrm{bond}}(i) >_T]_R$, $[< k_{\mathrm{bond}} >_T]_R$, and $[< k^2_{\mathrm{bond}} >_T]_R$ are shown on the Nishimori line. They are rigorous even on a finite number of nodes.

It is known that the internal energy, the upper bound of the specific heat, and so forth are exactly calculated on the Nishimori line without the dependence of the network (lattice).[5–9] In the present article, it is realized that, as a property for the Nishimori line, the bond does not depend on the size of the degree $k(i)$ since the bias for the size of the degree $k(i)$ does not appear in the statistical results of the bonds put between spins.

In the present article, the percolation thresholds are shown. It is anticipated that ineq. (22) is true for sufficiently large number of nodes in the case that a condition that the bond does not depend on the size of $k(i)$ is satisfied. The condition is given in the present article. Ineq. (23) is derived from ineq. (22) and is the criterion of the percolation of cluster for spin models. Ineq. (23) gives the percolation threshold in the case that the condition is satisfied. The percolation thresholds are calculated based on ineq. (23).

In the result for the Gaussian model, an approximation formula of the error function, eq. (65), is used. In the result for the Gaussian model, it is necessary for the more precise estimation of the percolation threshold that the error function in eq. (64) is numerically estimated.

It is shown that the percolation transition temperature $T_P^{(\pm J)}$ (eq. (32)) on the Nishimori line for the $\pm J$ Ising model on the present network coincides with the phase transition temperature $T_C^{(\mathrm{Ferro})}$[26,27] for the ferromagnetic Ising model on the same network. For the ferromagnetic Ising model on the same network, the specific heat diverges or jumps at $T_C^{(\mathrm{Ferro})}$.[26,27] On the other hand, from eq. (36), it is shown that the specfic heat neither diverge nor jump at $T_P^{(\pm J)}$ on the Nishimori line.

The results for the infinit-range $\pm J$ model and the results for the SK model are also shown. In Ref.,[18] the percolation transition temperature of the FK cluster for the infinit-range $\pm J$ model is derived by using the analytical solution of the SK model. The percolation transition temperature of the FK cluster for the infinit-range $\pm J$ model obtained in the present article agrees with the result for single replica case in Ref.[18]

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